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BOUNDARY RESONANCE IN A SEMI-INFINITE ELASTIC RIGIDLY FIXED WAVEGUIDE†

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On the basis of a numerical analysis, a boundary resonance in a semi-infinite elastic waveguide with mixed boundary conditions, when a part of the surface is rigidly fixed while a force which is harmonically dependent on time is imposed on the other part, is found and analysed. Here, the frequency found for the boundary resonance lies below the first stopping frequency of the waveguide so that the boundary resonance in the situation being considered is of the same nature as a resonance in a system without damping: an unlimited increase is observed in the amplitudes of the characteristics of the wave field as the frequency of the vibrations tends to a value Ω_e, unlike the conventional form of a resonance with finite amplitudes [1–3]. The specific behaviour of the amplitude of excitation of a normal wave with a purely imaginary propagation constant in the neighbourhood of the frequency of the boundary resonance Ω_e is found.

1. The problem of forced harmonic vibrations of an elastic semi-infinite waveguide $0<X<\infty$, $0<Z<2H$ with an unstressed end $X=0$, $0\leqslant Z\leqslant 2H$ and a rigidly fixed lower face $Z=0$, $X>0$ is considered. The field is excited by a normal force applied to the upper face on which it is assumed that there is no tangential stress. Hence, the boundary value problem for the displacement vector $\mathbf{u}=\{u_x(x,z),\ u_z(z,z)\}$ in the dimensionless coordinates $x=X/H$, $z=Z/H$ (the time factor is omitted everywhere) has the form

$$\begin{gathered}G\nabla^2\mathbf{u}+G(1-2\nu)^{-1}\operatorname{grad}\operatorname{div}\mathbf{u}+\rho\omega^2\mathbf{u}=0 \\ u_x(x,0)=u_z(x,0)=0,\quad x\geqslant 0 \\ \sigma_x(0,z)=\tau_{zx}(0,z)=0,\quad z\in[0,2] \\ \sigma_z(x,2)=2Gf(x),\quad \tau_{zx}(x,2)=0,\quad x\geqslant 0\end{gathered}\tag{1.1}$$

where G is the shear modulus of the material, ν is Poisson's ratio, ρ is the specific density and, according to Hooke's law, the stress field has the form

$$\frac{\sigma_x}{2G}=\frac{\partial u_x}{\partial x}+\frac{\nu}{1-2\nu}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right),\quad \frac{\sigma_z}{2G}=\frac{\partial u_z}{\partial z}+\frac{\nu}{1-2\nu}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_z}{\partial z}\right)$$

$$\frac{\tau_{zx}}{2G}=\frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)\tag{1.2}$$

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In addition to (1.1), it is assumed that the condition for the energy density to be integrable in the neighbourhoods of the corner points of the waveguide is satisfied. Next, we consider a range of frequencies up to the first cutoff frequency of the waveguide so that the displacement vector from (1.1) may be considered as being decaying as $x \to \infty$.

We will construct the solution of the boundary value problem (1.1) using the method of superposition [1, 4]. Omitting the intermediate calculations, we can represent the displacement vector in the form (everywhere below, summation over n is carried out from 0 to ∞):

$$
\begin{gathered}
u_x(x,z) = \frac{2}{\pi}\sum M_n \int_0^\infty \frac{q_1 h_{1n}(\tau) P_1(\tau,z) + (-1)^n h_{2n}(\tau) P_2(\tau,z)}{\Delta(\tau)} \cos(\tau x)\,d\tau - \\
- \sum M_n \left(p_1 \exp(-p_1 x) - \frac{\theta(\beta_n)}{p_2}\exp(-p_2 x)\right)\cos(\beta_n z) + \\
+ 2\int_0^\infty \frac{\tau F(\tau) P_3(\tau,z)}{\Delta(\tau)}\cos(\tau x)\,d\tau \\
u_z(x,z) = \frac{2}{\pi}\sum M_n \int_0^\infty \frac{q_1 h_{1n}(\tau) R_1(\tau,z) + (-1)^n h_{2n}(\tau) R_2(\tau,z)}{\Delta(\tau)} \sin(\tau x)\,d\tau - \\
- \sum M_n \left(\beta_n \exp(-p_1 x) - \frac{\theta(\beta_n)}{\beta_n}\exp(-p_2 x)\right)\sin(\beta_n z) + \\
+ 2\int_0^\infty \frac{q_1 F(\tau) R_3(\tau,z)}{\Delta(\tau)}\sin(\tau x)\,d\tau \\
p_j = p_j(n) \doteq \begin{cases} \sqrt{\beta_n^2 - \Omega_j^2}, & \beta_n \geqslant \Omega_j, \\ -i\sqrt{\Omega_j^2 - \beta_n^2}, & \beta_n < \Omega_j, \end{cases} \\
q_j = q_j(\tau) = \begin{cases} \sqrt{\tau^2 - \Omega_j^2}, & \tau \geqslant \Omega_j \\ -i\sqrt{\Omega_j^2 - \tau^2}, & 0 < \tau < \Omega_j \end{cases} \\
\Omega_j = \frac{\omega H}{c_j}, \quad j = 1, 2, \quad \beta_n = \frac{\pi(2n-1)}{4}, \quad n = 1, 2, \ldots \\
\theta(\xi) = \xi^2 - \frac{\Omega_2^2}{2}, \quad \xi \in \{\tau, \gamma, \beta_n\} \\
\Delta(\tau) = q_1 q_2(\tau^4 + \theta^2(\tau))\,\mathrm{ch}\,2q_1\,\mathrm{ch}\,2q_2 - \tau^2(q_1^2 q_2^2 + \theta^2(\tau))\,\mathrm{sh}\,2q_1\,\mathrm{sh}\,2q_2 - \\
- 2\tau^2 q_1 q_2 \theta(\tau) \\
P_j(\tau,z) = \mathrm{ch}\,q_2\,a_{1+j}(\tau) f_1(\tau,z) + \mathrm{sh}\,q_2\,a_{4+j}(\tau) f_2(\tau,z), \quad j = 1, 2 \\
P_3(\tau,z) = \{\mathrm{ch}\,q_1 z \Delta(\tau) + q_1\theta(\tau)\,\mathrm{ch}\,2q_1\,[\mathrm{ch}\,q_2 a_1(\tau) f_1(\tau,z) + \mathrm{sh}\,q_2 a_4(\tau)\cdot \\
\cdot f_2(\tau,z)]\}/(\theta(\tau)\,\mathrm{ch}\,2q_1) \\
R_j(\tau,z) = \mathrm{ch}\,q_2 a_{1+j}(\tau) d_1(\tau,z) + \mathrm{sh}\,q_2 a_{4+j}(\tau) d_2(\tau,z), \quad j = 1, 2 \\
R_3(\tau,z) = \{\mathrm{sh}\,q_1 z \Delta(\tau) + \tau\theta(\tau)\,\mathrm{ch}\,2q_1\,[\mathrm{ch}\,q_2 a_1(\tau) d_1(\tau,z) + \mathrm{sh}\,q_2 a_4(\tau)\cdot \\
\cdot d_2(\tau,z)]\}/(\theta(\tau)\,\mathrm{ch}\,2q_1) \\
F(\tau) = \frac{1}{\pi}\int_0^\infty f(x)\sin(\tau x)\,dx \\
a_1(\tau) = \theta(\tau)\,\mathrm{sh}\,q_2 + \tau^2\,\mathrm{sh}\,q_2\,\mathrm{ch}\,2q_1 - q_1 q_2\,\mathrm{ch}\,q_2\,\mathrm{sh}\,2q_1
\end{gathered}
\tag{1.3}
$$

$$a_2(\tau) = \tau^2 q_1 q_2 \operatorname{sh} 2q_1 \operatorname{ch} q_2 - \theta^2(\tau) \operatorname{ch} 2q_1 \operatorname{sh} q_2 - \tau^2\theta(\tau) \operatorname{sh} q_2$$

$$a_3(\tau) = -\tau^2\theta(\tau) \operatorname{sh} 2q_1 \operatorname{sh} q_2 - \tau^2 q_1 q_2 \operatorname{ch} q_2 + q_1 q_2 \theta(\tau) \operatorname{ch} 2q_1 \operatorname{ch} q_2$$

$$a_4(\tau) = \tau^2 \operatorname{ch} 2q_1 \operatorname{ch} q_2 - q_1 q_2 \operatorname{sh} 2q_1 \operatorname{sh} q_2 - \theta(\tau) \operatorname{ch} q_2$$

$$a_5(\tau) = \theta^2(\tau) \operatorname{ch} q_2 \operatorname{ch} 2q_1 - \tau^2 q_1 q_2 \operatorname{sh} 2q_1 \operatorname{sh} q_2 - \tau^2\theta(\tau) \operatorname{ch} q_2$$

$$a_6(\tau) = \tau^2 q_1 q_2 \operatorname{sh} q_2 + q_1 q_2 \theta(\tau) \operatorname{sh} q_2 \operatorname{ch} 2q_1 - \tau^2\theta(\tau) \operatorname{sh} 2q_1 \operatorname{ch} q_2$$

$$h_{1n}(\tau) = \frac{\tau^2(\Omega_2^2 - 2\Omega_1^2) - \beta_n^2\Omega_2^2 + \Omega_1^2\Omega_2^2}{2(\tau^2 + \beta_n^2 - \Omega_1^2)(\tau^2 + \beta_n^2 - \Omega_2^2)}$$

$$h_{2n}(\tau) = (-1)^n \frac{4\tau^2\beta_n^2(\Omega_2^2 - \Omega_1^2) - (\tau^2 + \beta_n^2)\Omega_2^4 + \Omega_1^2\Omega_2^4}{4\beta n(\tau^2 + \beta_n^2 - \Omega_1^2)(\tau^2 + \beta_n^2 - \Omega_2^2)}$$

$$f_1(\tau, z) = -\frac{\tau^2 \operatorname{sh} q_1(z-2)}{q_1 \operatorname{ch} 2q_1} - \frac{\tau^2 q_2 \operatorname{sh} q_2 \operatorname{ch} q_1 z}{\theta(\tau) \operatorname{ch} q_2 \operatorname{ch} 2q_1} + q_2 \frac{\operatorname{sh} q_2(z-1)}{\operatorname{ch} q_2}$$

$$f_2(\tau, z) = \frac{\tau^2 \operatorname{sh} q_1(z-2)}{q_1 \operatorname{ch} 2q_1} - \frac{\tau^2 q_2 \operatorname{ch} q_2 \operatorname{ch} q_1 z}{\theta(\tau) \operatorname{sh} q_2 \operatorname{ch} 2q_1} + q_2 \frac{\operatorname{ch} q_2(z-1)}{\operatorname{sh} q_2}$$

$$d_1(\tau, z) = -\frac{\tau \operatorname{ch} q_1(z-2)}{\operatorname{ch} 2q_1} - \frac{\tau q_1 q_2 \operatorname{sh} q_2 \operatorname{sh} q_1 z}{\theta(\tau) \operatorname{ch} q_2 \operatorname{ch} 2q_1} + \tau \frac{\operatorname{ch} q_2(z-1)}{\operatorname{ch} q_2}$$

$$d_2(\tau, z) = \frac{\tau \operatorname{ch} q_1(z-2)}{\operatorname{ch} 2q_1} - \frac{\tau q_1 q_2 \operatorname{ch} q_2 \operatorname{sh} q_1 z}{\theta(\tau) \operatorname{sh} q_2 \operatorname{ch} 2q_1} + \tau \frac{\operatorname{sh} q_2(z-1)}{\operatorname{sh} q_2}$$

Here, $c_1 = (2G(1-\nu)/((\rho(1-2\nu)))^{1/2}$, $c_2 = (G/\rho)^{1/2}$ are the velocities of the longitudinal and transverse waves in the waveguide material and M_n ($n = 1, 2, \ldots$) is a sequence of unknown numbers. Here, $\Delta(\tau) = 0$ is the dispersion equation for an infinite layer $|x| < \infty$, $0 < z < 2$ with a rigidly fixed lower face $z = 0$ and an unstressed face $z = 2$ [4]. The displacement vector in the form of (1.3) satisfies all the boundary conditions from (1.1) with the exception of the condition $\tau_{zx}(0, z) = 0$ (regardless of the dependence on the values of the coefficients M_n). Satisfaction of this remaining boundary condition leads to the following infinite system of linear algebraic equations in the unknowns M_n:

$$M_n = \frac{2p_2\beta_n}{\theta^2(\beta_n) - p_1 p_2 \beta_n^2} \left\{ (-1)^{n-1} \int_0^\infty \frac{\tau q_1^2 F(\tau)\, d\tau}{\theta(\tau)(q_1^2 + \beta_n^2)} + \right.$$

$$+ (-1)^n \int_0^\infty [C(\tau) \operatorname{th} q_2 + D(\tau) \operatorname{cth} q_2]\, a_{31}(\tau, n)\, d\tau +$$

$$\left. + \beta_n \int_0^\infty [C(\tau) - D(\tau)]\, a_{32}(\tau, n)\, d\tau \right\}$$

$$C(\tau)\Delta(\tau) = \tau q_1 F(\tau) \operatorname{ch} q_2 a_1(\tau) + q_1 S_{1n}(\tau) \operatorname{ch} q_2 a_2(\tau) + S_{2n}(\tau) \operatorname{ch} q_2 a_3(\tau)$$

$$D(\tau)\Delta(\tau) = \tau q_1 F(\tau) \operatorname{sh} q_2 a_4(\tau) + q_1 S_{1n}(\tau) \operatorname{sh} q_2 a_5(\tau) + S_{2n}(\tau) \operatorname{sh} q_2 a_6(\tau) \qquad (1.4)$$

$$S_{1n}(\tau) = \frac{1}{\pi} \sum M_n \left(\frac{p_1^2}{\tau^2 + p_1^2} - \frac{\theta(\beta_n)}{\tau^2 + p_2^2} \right)$$

$$S_{2n}(\tau) = \frac{1}{\pi} \sum (-1)^n M_n \left(\frac{p_1^2 \beta_n}{\tau^2 + p_1^2} - \frac{\theta^2(\beta_n)}{\beta_n(\tau^2 + p_2^2)} \right)$$

$$a_{11}(\tau) = \tau^2 \frac{\operatorname{th} 2q_1}{q_1} - \frac{\tau^2 q_2 \operatorname{th} q_2}{\theta(\tau) \operatorname{ch} 2q_1} - q_2 \operatorname{th} q_2$$

$$a_{12}(\tau) = -\tau^2 \frac{\operatorname{th} 2q_1}{q_1} - \frac{\tau^2 q_2 \operatorname{cth} q_2}{\theta(\tau) \operatorname{ch} 2q_1} + q_2 \operatorname{cth} q_2$$

$$a_{21}(\tau)=\theta(\tau)-\frac{\tau^2 q_1 q_2 \operatorname{th} 2q_1 \operatorname{th} q_2}{\theta(\tau)}-\frac{\tau^2}{\operatorname{ch} 2q_1}$$

$$a_{22}(\tau)=\theta(\tau)-\frac{\tau^2 q_1 q_2 \operatorname{th} 2q_1 \operatorname{cth} q_2}{\theta(\tau)}+\frac{\tau^2}{\operatorname{ch} 2q_1}$$

$$a_{31}(\tau, n)=q_2\left(\frac{\tau^2 q_1^2}{\theta(\tau)(q_1^2+\beta_n^2)}-\frac{\theta(\tau)}{q_2^2+\beta_n^2}\right)$$

$$a_{32}(\tau, n)=\frac{\theta(\tau)}{q_2^2+\beta_n^2}-\frac{\tau^2}{q_1^2+\beta_n^2}$$

In order to take account of the contribution of all of the unknowns M_n of the infinite system (1.4) during its reduction to a finite algebraic system of equations in M_n ($n = 1, \ldots, N$) and, also, for a full analysis within the framework of the approach of the immediate wave field in the neighbourhood of an end of the waveguide which has been adopted, *a priori* knowledge of the asymptotic behaviour of the unknowns M_n when $n \to \infty$ is of decisive importance. An investigation of system (1.4) using the technique of the Mellin integral transform (see [5]) shows that the asymptotic form

$$M_n = a\beta_n^{-\gamma} + b(-1)^n \beta_n^{-1} + O(\beta_n^{-2}), \quad n \to \infty \tag{1.5}$$

holds in the case of a bounded solution $\{M_n\}_n^{\infty=1}$ of system (1.4), where a and b are certain constants which are linearly dependent on the specified load $f(x)$ and $\gamma = \gamma(\nu) \in (1/2, 1)$ is the first root, in ascending order of the real parts, of the equation

$$(3-4\nu)\cos^2\frac{1}{2}\pi\gamma + (1-2\nu)^2 - \gamma^2 = 0, \quad \operatorname{Re}\gamma > 0 \tag{1.6}$$

Here, the second term on the right-hand side of (1.5) corresponds to the first root of the equation $\sin^2(\pi\gamma/2) - \gamma^2 = 0$ in the same half plane $\operatorname{Re}\gamma > 0$.

As regards the organization of the computational process for solving the truncated system (1.4) and the qualitative and quantitative analysis of the stressed-deformed state of the waveguide in the neighbourhoods of the corner points on the basis of the use of the asymptotic behaviour of M_n in (1.4) and (1.3) respectively, see [1, 5].

2. The boundary value problem (1.1) was solved numerically on the basis of representation (1.3) and system (1.4) for the load

$$f(x)=\begin{cases}1, & x \in [^1/_2, 1] \\ 0, & x \overline{\in} [^1/_2, 1]\end{cases} \tag{2.1}$$

for Poisson's ratio $\nu = 0.3$ over a range of frequencies $0.4 \leqslant \Omega_2 \leqslant 0.78$ so that $\Omega_2 < \pi/4 = \Omega^{(1)}$, where $\Omega^{(1)}$ is the first cutoff frequency of the waveguide under consideration, that is, there are no travelling waves in the frequency range being considered. This investigation showed that, as the frequency Ω_2 approaches $\Omega_e = 0.761$, a sharp increase is observed in the absolute magnitudes of the coefficients M_n ($n = 1, \ldots, 10$) which are the solution of the finite algebraic system obtained by the truncation of system (1.4) using the asymptotic form (1.5). At the same time, on passing from a frequency $\Omega_2 = 0.761$ to the frequency $\Omega_2 = 0.7615$, there is a change in the sign of all the unknowns and the sign of the frequency determinant of the system. Upon becoming more remote from the frequency $\Omega_2 = 0.7615$, the absolute magnitudes of the unknown M_n decrease and the accuracy to which boundary condition (1.1) is satisfied increases. These special features are the attributes of boundary resonance [1].

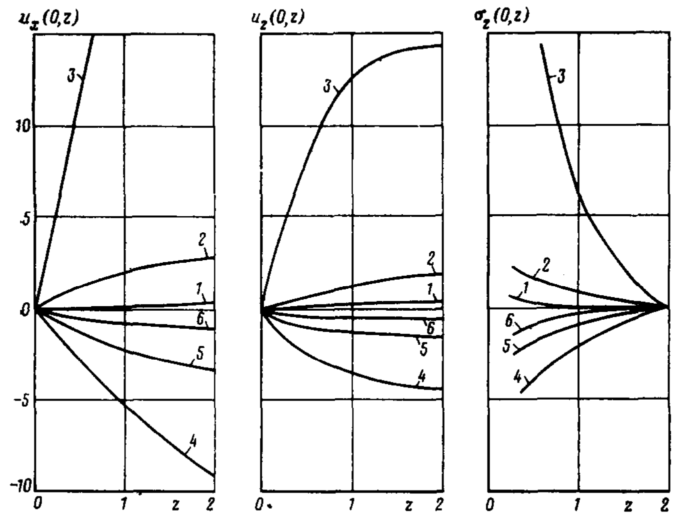

FIG. 1.

The displacements $u_x(0, z)$ and $u_z(0, z)$ and the stress $\sigma_z(0, z)$ on the end $x = 0$ of the waveguide are shown in Fig. 1 for different frequencies. Curves 1–6 correspond to values of the frequency $\Omega_2 = 0.4$, 0.75, 0.76, 0.765, 0.77, 0.78, respectively. When $\Omega_2 = 0.4$ and $\Omega_2 = 0.78$, the displacements at the end are practically equal to zero. As the frequency approaches the boundary resonance $\Omega_e = 0.761$, the absolute magnitudes of the displacements and the stress $\sigma_z(0, z)$ at the end increase and, on passing through the frequency of the boundary resonance, the displacements and the stress change sign and, as the frequency recedes from Ω_e, they decrease sharply in absolute magnitude.

The displacements of the segment $x \in [0, 4]$ of the free face are shown in Fig. 2(a,b) for the same frequencies as in Fig. 1. Here, a pronounced localization of the motion is observed near the end at around a frequency of Ω_e and a sharp fall-off in the displacements on moving away from the end. The displacements change sign on passing through the frequency Ω_e. The characteristic bumps in Fig. 2 correspond to displacements on the segment of the surface $x \in [1/2, 1]$ under the specified load. The behaviour of the curves in Fig. 2 clearly shows the possibility of obtaining reliable data on the characteristic mode of the vibrations of a waveguide when forced motions are considered. Whereas, for small Ω_2, the deflection follows behind the load (curves 1, 2), at Ω_2 close to the resonance frequency the motion of the surface is determined by the characteristic mode and is only slightly distorted by the external load.

The above-mentioned features of a boundary resonance in a semi-infinite waveguide with a rigidly fixed face are analogous to the phenomena of boundary resonance accompanying the longitudinal vibrations of a free semi-infinite waveguide, excited by a load at the end in the case when $\nu = 0$ [6] and, also, in the case of vibrations of a long cylinder with a fixed lateral surface [7].

3. Since the frequency $\Omega_e \in (0, \Omega^{(1)})$ in the situation being considered, it is of interest to trace the contribution of each inhomogeneous wave to the formation of the wave field close to the end of the waveguide. The solution of the boundary value problem (1.1) in the form of (1.3) can be re-expanded in the form of a series in the homogeneous solutions for a corresponding layer. Namely [4], in the case of a load (2.1), the displacement vector when $x \in (0, 1/2)$ is represented in the form

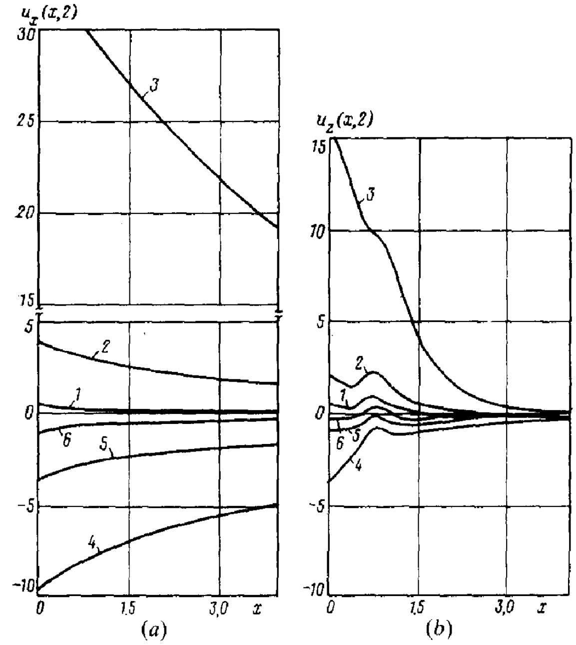

FIG. 2.

$$u(x, z) = \Sigma c_k^+ \mathbf{u}_{\gamma_k^+}(z) \exp(i\gamma_k^+ x) + \Sigma c_k^- \mathbf{u}_{\gamma_k^-}(z) \exp(i\gamma_k^- x)$$

$$\mathbf{u}_\gamma(z) = \{u_x(\gamma, z),\quad u_z(\gamma, z)\},$$

$$u_x(\gamma, z) = [q_1 q_2 N(\gamma)(\operatorname{ch} q_2 z - \operatorname{ch} q_1 z) - M(\gamma)(\gamma^2 \operatorname{sh} q_1 z - q_1 q_2 \operatorname{sh} q_2 z)]/ /(\gamma M(\gamma))$$

$$u_z(\gamma, z) = -iq_1[\gamma^2 M(\gamma)(\operatorname{ch} q_2 z - \operatorname{ch} q_1 z) + N(\gamma)(\gamma^2 \operatorname{sh} q_2 z - - q_1 q_2 \operatorname{sh} q_1 z)]/(\gamma^2 M(\gamma)) \tag{3.1}$$

$$M(\gamma) = \theta(\gamma) \operatorname{sh} 2q_2 - q_1 q_2 \operatorname{sh} 2q_1, \quad N(\gamma) = \gamma^2 \operatorname{ch} 2q_1 - - \theta(\gamma) \operatorname{ch} 2q_2$$

$$c_k^+ = 2\pi i T(\tau) \left. \frac{S_{1n}(\tau) d_{21}(\tau) - S_{2n}(\tau) d_{11}(\tau) + \tau q_1 F^+(\tau) d_{33}(\tau)}{d_{11}(\tau) \Delta'(\tau)} \right|_{\tau = \gamma_k^+}$$

$$c_k^- = -2\pi i \left. \frac{\tau q_1 F^-(\tau) d_{33}(\tau)}{d_{11}(\tau) \Delta'(\tau)} \right|_{\tau = \gamma_k^-}$$

$$T(\tau) = \tau q_2 (\tau^2 \operatorname{ch} 2q_2 - \theta(\tau) \operatorname{ch} 2q_1)$$

$$F^{\pm}(\tau) = \frac{1}{2\pi i} \int_0^\infty f(x) \exp(\pm i\tau x)\, dx.$$

Representation (3.1), with the coefficients $c_k^{\pm}$ expressed in terms of the sequence M_n, enables one to obtain quantitative estimates of the coefficients for the excitation of the normal modes as a function of frequency.

The absolute magnitudes of the normalized amplitudes of the normal waves ($d_k = c_k/|u_2^k(0, 1)|$) are shown in Fig. 3 as a function of frequency. Curve 1 is a wave with a purely imaginary

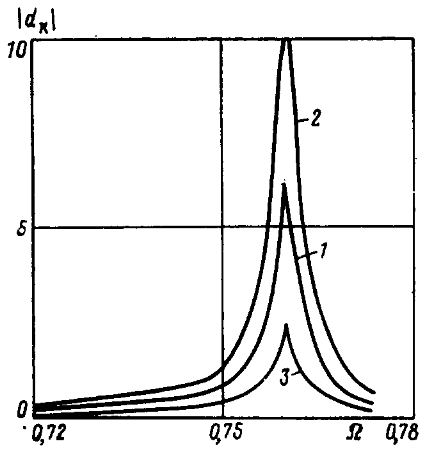

FIG. 3.

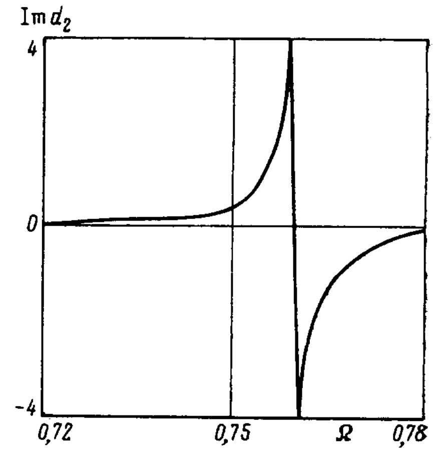

FIG. 4.

propagation constant and curves 2 and 3 are waves with the first and second complex propagation constants, respectively. In a narrow range of frequencies close to Ω_e, a sharp increase in the normalized amplitudes of all of the inhomogeneous waves is observed. As in the case of a waveguide with free lateral faces, the inhomogeneous wave with the first complex propagation constant is the most strongly excited and is decisive in the formation of the wave field at the end face of the waveguide. Unlike the case of a boundary resonance in a free waveguide, it manifests itself in the presence of a substantial contribution from the wave with the purely imaginary propagation constant. A similar phenomenon has been observed in the analysis of the vibrational modes of a cylinder with a fixed lateral surface [7].

The change in the phase of the normalized amplitude of the wave with the first complex propagation constant on passing through the frequency of the marginal resonance Ω_e is shown in Fig. 4.

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